GEVREY REGULARITY OF SUBELLIPTIC MONGE-AMPÈRE EQUATIONS IN THE PLANE

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ABSTRACT. In this paper, we establish the Gevrey regularity of solutions for a class of degenerate Monge-Ampère equations in the plane, under the assumption that one principle entry of the Hessian is strictly positive and an appropriately finite type degeneracy.

1. INTRODUCTION

In this paper, we study the regularity problem for the real Monge-Ampère equation

$$\det\left(D^2u\right) = k(x), \quad x \in \Omega \subset \mathbb{R}^d, \tag{1.1}$$

where Ω is an open domain of $\mathbb{R}^d, d \geq 2$. We consider the convex solution u of equation (1.1), then k is a nonnegative function. In the case when $k > 0$, the equation (1.2) is elliptic and the theory is well developed. For instance, it's shown in [1] that there exists a unique solution u to the Dirichlet problem for (1.1), smooth up to the boundary of Ω, provided that k is smooth and the boundary $\partial\Omega$ of Ω is strictly convex. In the degenerate case, i.e.,

$$\Sigma_k = \{x \in \Omega; k(x) = 0, \nabla k(x) = 0\} \neq \emptyset.$$

The equation (1.1) is then a full nonlinear degenerate elliptic equation. The existence and uniqueness of the solution for the Dirichlet problem of the equation (1.1) have already been studied in [10]. Also in [12], they proved that the Monge-Ampère equation has a C^∞ convex local solution if the order of degenerate point for the smooth coefficient k is finite.

As far as the regularity problem is concerned, a result in [19] proved that, for the degenerate Monge-Ampère equation, if the solution $u \in C^\rho$ for $\rho > 4$ (so that it is a classical solution), then u will be C^∞ smooth.

However, in general, the convex solution u to (1.1) is at most in $C^{1,1}$ if k is only smooth and nonnegative (see [8] for example). To get a higher regularity, some extra assumptions are needed to impose on k. This problem has been studied by P. Guan [9] in two dimension case, in which the smoothness of C^∞ for a $C^{1,1}$ solution u of the equation (1.1) is obtained, if k vanishes in finite order, i.e. $k \approx x^{2\ell} + Ay^{2n}$ with $\ell \leq n, A \geq 0$, and one principal curvature of u is strictly positive. In a recent paper [11], the last assumption is relaxed to the bounding of trace of Hessian from below, i.e., $\triangle u \geq c_0 > 0$. For such C^∞ regularity problem, see also earlier work of C.-J. Xu [18] which is concerned with the C^∞ regularity for general two-dimensional degenerate elliptic equation. In a recent work [13], the authors extended Guan's two-dimensional result of [9] to higher dimensional case.

It is natural to ask that, in the degenerate case, would it be the best possible for the regularity of solution here to be C^∞ smooth? One may expect that, in case of coefficient

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k with higher regularity, the solution u would have better regularity than C^∞ smooth. we will introduce the Gevrey class, an intermediate space between the spaces of the analytic functions and the C^∞ functions. There is well-developed theory on the Gevrey regularity (see the definition later) for nonlinear elliptic equations of any order, see [7] for instance. For the linear degenerate elliptic problem, there have been many works on the Gevrey hypoellipticity of linear subelliptic operators of second order (e.g. [4, 5] and the reference therein). The difficulty concerned with equation (1.1) lies on the mixture of degeneracy and nonlinearity.

In this paper, we attempt to explore the regularity of solutions of equation (1.1) in the frame of Gevrey class. We study the problem in two dimension case

$$u_{xx}u_{yy} - u_{xy}^2 = k(x,y), \quad (x,y) \in \Omega, \tag{1.2}$$

and assume that $u_{yy} > 0$, then we can apply the classic partial Legendre transformation (see [16] for instance), to translate the equation (1.2) to the following divergence form quasi-linear equation

$$\partial_{ss} w(s,t) + \partial_t \{k(s, w(s,t))\partial_t w(s,t)\} = 0. \tag{1.3}$$

This quasi-linearity allows us to adopt the idea used in [2], to obtain the Gevrey regularity for the above divergence form equation. In order to go back to the original problem, i.e., the Gevrey regularity for the equation (1.2), a key point would be to show that the Gevrey regularity is invariant under the partial Legendre transformation, which will be proved in Section 3.

Now let us recall the definition of the space of Gevrey class functions, which is denoted by $G^\sigma(U)$, for $\sigma \geq 1$, with U an open subset of $\mathbb{R}^d$ and σ being called Gevrey index. We say that $f \in G^\sigma(U)$ if $f \in C^\infty(U)$ and for any compact subset K of U, there exists a constant C_K, depending only on K, such that for all multi-indices $\alpha \in \mathbb{Z}_+^d$,

$$\|\partial^\alpha f\|_{L^\infty(K)} \leq C_K^{|\alpha|+1}(|\alpha|!)^\sigma.$$

The constant C_K here is called the Gevrey constant of f. We remark that the above inequality is equivalent to the following condition:

$$\|\partial^\alpha f\|_{L^2(K)} \leq C_K^{|\alpha|+1}(|\alpha|!)^\sigma.$$

In this paper, both estimates above will be used. Observe that $G^1(U)$ is the space of real analytic functions in U.

We state now our main result as follows, where Ω is an open neighborhood of origin in $\mathbb{R}^2$.

Theorem 1.1. *Let u be a $C^{1,1}$ weak convex solution to the Monge-Ampère equation (1.2). Suppose that $u_{yy} \geq c_0 > 0$ in Ω and that $k(x,y)$ is a smooth function defined in Ω, satisfying*

$$c^{-1}(x^{2\ell} + A y^{2n}) \leq k(x,y) \leq c(x^{2\ell} + A y^{2n}), \qquad (x,y) \in \Omega \tag{1.4}$$

where $c > 0, A \geq 0$ and $\ell \leq n$ are two nonnegative integers. Then $u \in G^{\ell+1}(\Omega)$, provided $k \in G^{\ell+1}(\Omega)$.

Remark 1.2. *If k is C^∞ smooth and satisfies the condition (1.4), and $u_{yy} > 0$, P. Guan* [9] *has proved that a $C^{1,1}$ solution of the equation (1.2) will be C^∞ smooth. In* [11], *the assumption that $u_{yy} > 0$ is relaxed to the bounding of trace of Hessian from below, i.e., $\triangle u \geq c_0 > 0$, but the assumption (1.4) is changed to $A > 0$ and $\ell = n$. Our main contribution here is to obtain the Gevrey regularity $G^{\ell+1}$.*

Remark 1.3. *The regularity result of main theorem seems the best possible, since in the particular case of $\ell = 0$, we have $u \in G^1(\Omega)$ (i.e., the solution is analytic in Ω), and in this case the equation (1.2) is elliptic, thus our result coincides with the well-known analytic regularity result for nonlinear elliptic equations. We can also justify that if k is independent of second variable y (then $A = 0$), the equation (1.3) is linear, it is known that, see* [4]*, the optimal regularity result is that the solution lies in $G^{\ell+1}$.*

Remark 1.4. *The extension of above result to higher dimensional cases and more general models of the Monge-Amère equations with $k = k(x, u, Du)$ is our coming work. By using the results of* [13]*, the idea is the same.*

The paper is organized as follows: the section 2 is devoted to proving the Gevrey regularity for the quasi-linear equation (1.3). In Section 3 we prove our main result by virtue of the classic partial Legendre transformation. We prove the technical lemmas in Section 4.

2. Gevrey regularity of quasi-linear subelliptic equations

In this section we study the Gevrey regularity of solutions for the following quasi-linear equation near the origin of $\mathbb{R}^2$

$$\partial_{ss} w + \partial_t \left(k(s, w)\partial_t w\right) = 0. \tag{2.1}$$

We assume that $k(s, w)$ satisfies the condition

$$c^{-1}(s^{2\ell} + Aw^{2n}) \leq k(s, w) \leq c(s^{2\ell} + Aw^{2n}), \tag{2.2}$$

where $c > 1, A \geq 0$ are two constants and $\ell \leq n$ are two positive integers. Since Gevrey regularity is a local property, we study the problem on the unit ball in $\mathbb{R}^2$,

$$B = \left\{(s, t) \;\middle|\; s^2 + t^2 < 1\right\},$$

and denote $W = [-1, 1] \times [-\|w\|_{L^\infty(\bar{B})}, \|w\|_{L^\infty(\bar{B})}]$. We prove the the following result in this section.

Theorem 2.1. *Suppose that $w(s, t) \in C^\infty(\bar{B})$ is a solution to the quasi-linear equation (2.1), and that $k \in G^{\ell+1}(W)$. Then $w \in G^{\ell+1}(B)$.*

We recall some notations and elementary results for the Sobolev space and pseudo-differential operators. Let $H^\kappa(\mathbb{R}^2), \kappa \in \mathbb{R}$, be the classical Sobolev space equipped with the norm $\|\cdot\|_\kappa$. Observe $\|\cdot\|_0 = \|\cdot\|_{L^2(\mathbb{R}^2)}$. Recall that $H^\kappa(\mathbb{R}^2)$ is an algebra if $\kappa > 1$. We need also the interpolation inequality for Sobolev space: for any $\varepsilon > 0$ and any $r_1 < r_2 < r_3$,

$$\|h\|_{r_2} \leq \varepsilon\|h\|_{r_3} + \varepsilon^{-(r_2-r_1)/(r_3-r_2)}\|h\|_{r_1}, \quad \forall\, h \in H^{r_3}(\mathbb{R}^2). \tag{2.3}$$

Let U be an open subset of $\mathbb{R}^2$ and $S^a(U), a \in \mathbb{R}$, be the symbol space of classical pseudo-differential operators. We say $P = P(s, t, D_s, D_t) \in \mathrm{Op}(S^a(U))$, a pseudo-differential operator of order a, if its symbol $\sigma(P)(s, t; \zeta, \eta) \in S^a(U)$ with (ζ, η) the dual variable of (s, t). If $P \in \mathrm{Op}(S^a(U))$, then P is a continuous operator from $H^\kappa_c(U)$ to $H^{\kappa-a}_{loc}(U)$. Here $H^\kappa_c(U)$ is the subspace of $H^\kappa(\mathbb{R}^2)$ consisting of the distributions having their compact support in U, and $H^{\kappa-a}_{loc}(U)$ consists of the distributions h such that $\phi h \in H^{\kappa-a}(\mathbb{R}^2)$ for any $\phi \in C^\infty_0(U)$. For more detail on the pseudo-differential operator, we refer to the book [17]. Remark that if $P_1 \in \mathrm{Op}(S^{a_1})$, $P_2 \in \mathrm{Op}(S^{a_2}(U))$, then

$[P_1,\ P_2] \in \mathrm{Op}(S^{a_1+a_2-1}(U))$. In this paper, we shall use the pseudo-differential operator $\Lambda^r = \left(1+|D_s|^2+|D_t|^2\right)^{\frac{r}{2}}$ of order $r, r\in\mathbb{R}$, whose symbol is given by

$$\sigma(\Lambda^r) = \left(1+\zeta^2+\eta^2\right)^{\frac{r}{2}}.$$

In the following discussions, we denote, for $P \in \mathrm{Op}(S^a)$,

$$\left\|P\partial^m v\right\|_\kappa = \sum_{|\alpha|=m}\left\|P\partial^\alpha v\right\|_\kappa \quad \text{and } [v]_{j,U} = \sum_{|\gamma|=j}\left\|\partial^\gamma v\right\|_{L^\infty(U)}.$$

We consider the following linearized operator corresponding to (2.1) and the solution w,

$$\mathcal{L} = \partial_{ss} + \partial_t(\tilde{k}(s,t)\partial_t\ \cdot\ ),$$

where $\tilde{k}(s,t) = k(s,w(s,t))$. To simplify the notation, we extended smoothly the function $\tilde{k}$ to $\mathbb{R}^2$ by constant outside of $\bar{B}$, similar for k. We have firstly the following subelliptic estimate.

Lemma 2.2. *Under the assumption (2.2), for any $r\in\mathbb{R}$, there exists $C_r>0$ such that*

$$||v||^2_{r+\frac{1}{\ell+1}} + \left\|\partial_s\Lambda^r v\right\|_0^2 + ||\tilde{k}^{\frac{1}{2}}\partial_t\Lambda^r v||_0^2 \le C_r\left\{||\mathcal{L}v||^2_{r-\frac{1}{\ell+1}} + ||v||_0^2\right\}, \tag{2.4}$$

for any $v\in C_0^\infty(B)$, where C_r depends only on $[\tilde{k}]_{j,\bar{B}}, 0\le j\le 2$.

Remark 2.3. *By using Faà di Bruno's formula, $[\tilde{k}]_{j,\bar{B}}$ is bounded by a polynomial of $[k]_{i,W}, [w]_{i,\bar{B}}, 0\le i\le j$.*

Proof. Firstly, we study the case of $r=0$. Observe

$$||\partial_s v||_0^2 + ||\tilde{k}^{\frac{1}{2}}\partial_t v||_0^2 = ||\partial_s v||_0^2 + \int_{\mathbb{R}^2}\tilde{k}(s,t)|\partial_t v(s,t)|^2 dsdt = -(\mathcal{L}v, v). \tag{2.5}$$

Then the assumption (2.2) implies

$$||\partial_s v||_0^2 + ||s^\ell\partial_t v||_0^2 \le c\{||\partial_s v||_0^2 + ||\tilde{k}^{\frac{1}{2}}\partial_t v||_0^2\} = -c(\mathcal{L}v, v).$$

Since the vector fields $\{\partial_s, s^\ell\partial_t\}$ satisfies the Hörmander's condition of order ℓ, we get (see [6, 15])

$$||v||^2_{\frac{1}{\ell+1}} \le C_0\{||\partial_s v||_0^2 + ||\tilde{k}^{\frac{1}{2}}\partial_t v||_0^2 + ||v||_0^2\} = -C_0(\mathcal{L}v, v) + C_0||v||_0^2. \tag{2.6}$$

By Cauchy-Schwarz inequality, we have proved (2.4) with $r=0$. Since we have extended $\tilde{k}$ to $\mathbb{R}^2$, (2.5) (2.6) also hold for any $v\in\mathcal{S}(\mathbb{R}^2)$.

Now for the general case, we have

$$\begin{aligned}||\partial_s\Lambda^r v||_0^2 + ||\tilde{k}^{\frac{1}{2}}\partial_t\Lambda^r v||_0^2 &= -(\Lambda^r\mathcal{L}v,\ \Lambda^r v) - \left([\tilde{k},\ \Lambda^r]\partial_t v,\ \partial_t\Lambda^r v\right)\\ &\le ||\mathcal{L}v||^2_{r-\frac{1}{\ell+1}} + ||v||^2_{r+\frac{1}{\ell+1}} - \left([\tilde{k},\ \Lambda^r]\partial_t v,\ \partial_t\Lambda^r v\right).\end{aligned}$$

Since for $v\in C_0^\infty(B)$, we have $\Lambda^r v\in\mathcal{S}(\mathbb{R}^2)$. Then (2.6) implies

$$\begin{aligned}&||v||^2_{r+\frac{1}{\ell+1}} + ||\partial_s\Lambda^r v||_0^2 + ||\tilde{k}^{\frac{1}{2}}\partial_t\Lambda^r v||_0^2\\ &\le C_0\left\{||\mathcal{L}v||^2_{r-\frac{1}{\ell+1}} + ||v||_0^2 - \left([\tilde{k},\ \Lambda^r]\partial_t v,\ \partial_t\Lambda^r v\right)\right\}.\end{aligned} \tag{2.7}$$

We consider now the commutator $[\tilde{k},\ \Lambda^r]$, the pseudo-differential calculus give

$$\sigma([\tilde{k},\ \Lambda^r]) = \sum_{|\alpha|=1}\partial^\alpha_{s,t}\tilde{k}(s,t)\partial^\alpha_{\zeta,\eta}\sigma(\Lambda^r)(\zeta,\eta) + \sigma(R_2)(s,t,\zeta,\eta),$$

with $\sigma(R_2) \in S^{r-2}(\mathbb{R}^2)$ and

$$\left|(R_2\partial_t v,\ \partial_t \Lambda^r v)\right| \le C_2 \|v\|_r^2,$$

where C_2 depends only on $[\,\tilde{k}\,]_{j,\bar{B}}, 0 \le j \le 2$. Thus

$$\left([\tilde{k},\ \Lambda^r]\partial_t v,\ \partial_t \Lambda^r v\right) \le C_0\,\|v\|_r \left\{\left\|\left(\partial_s \tilde{k}\right)\partial_t \Lambda^r v\right\|_0 + \left\|\left(\partial_t \tilde{k}\right)\partial_t \Lambda^r v\right\|_0\right\} + C\|v\|_r^2.$$

Moreover, note that $\tilde{k}$ is nonnegative, and hence we have the following well-known inequality

$$|\partial_s \tilde{k}(s,t)|^2 + |\partial_t \tilde{k}(s,t)|^2 \le 4[\,\tilde{k}\,]_{2,\mathbb{R}^2}\tilde{k}(s,t). \tag{2.8}$$

For the sake of completeness, we will present the proof of the above inequality later. By Cauchy-Schwarz inequality and interpolation inequality (2.3), one has

$$\left([\tilde{k},\ \Lambda^r]\partial_t v,\ \partial_t \Lambda^r v\right) \le \frac{1}{2C_0}\Big(\left\|\tilde{k}^{\frac{1}{2}}\partial_t \Lambda^r\ v\right\|_0^2 + \|v\|^2_{r+\frac{1}{\ell+1}}\Big) + C_r\|v\|_0^2.$$

Thus Lemma 2.2 follows. Now it remains to show (2.8). For any $h \in \mathbb{R}$, the following formula holds

$$\tilde{k}(s+h,t) = \tilde{k}(s,t) + \partial_s \tilde{k}(s,t)h + \frac{1}{2}\partial_{ss}\tilde{k}(s_0,t)h^2, \quad s_0 \in \mathbb{R}.$$

Observe $\tilde{k} \ge 0$, then for all $h \in \mathbb{R}$ we get $0 \le \tilde{k}(s,t) + \partial_s \tilde{k}(s,t)h + \frac{1}{2}[\,\tilde{k}\,]_{2,\mathbb{R}^2}h^2$. So the the discriminant of this polynomial is nonpositive; that is,

$$|\partial_s \tilde{k}(s,t)|^2 \le 2[\,\tilde{k}\,]_{2,\mathbb{R}^2}\tilde{k}(s,t).$$

Similarly $|\partial_t \tilde{k}(s,t)|^2 \le 2[\,\tilde{k}\,]_{2,\mathbb{R}^2}\tilde{k}(s,t)$. This gives (2.8). □

Remark 2.4. *With same proof, we can also prove the following estimate*

$$\|v\|^2_{r+m+\frac{1}{\ell+1}} + \sum_{|\alpha|\le m}\left(\left\|\partial_s \Lambda^r \partial^\alpha v\right\|_0^2 + \|\tilde{k}^{\frac{1}{2}}\partial_t \Lambda^r \partial^\alpha v\|_0^2\right) \le C_{r,m}\left\{\|\mathcal{L}v\|^2_{r+m-\frac{1}{\ell+1}} + \|v\|_0^2\right\}$$

for any $v \in C_0^\infty(B)$.

A key technical step in the proof of Gverey regularity is to choose a adapted family of cutoff functions. For $0 < \rho < 1$, set

$$B_\rho = \left\{(s,t)\ \middle|\ s^2 + t^2 < 1 - \rho\right\}.$$

For each integer $m \ge 2$ and each number $0 < \rho < 1$, we choose the cutoff function $\varphi_{\rho,m}$ satisfying the following properties:

$$\begin{cases} \operatorname{supp}\varphi_{\rho,m} \subset B_{\frac{(m-1)\rho}{m}}, \quad \text{and } \varphi_{\rho,m}(s,t) = 1 \text{ in } B_\rho, \\ \sup\limits_{(s,t)\in B}\left|\partial^k \varphi_{\rho,m}\right| \le C_k\left(\frac{m}{\rho}\right)^k. \end{cases} \tag{2.9}$$

For such cut-off functions, we have the following

Lemma 2.5 (Corollary 0.2.2 of [5])**.** *There exists a constant C, such that for any $0 \le \kappa \le 4$, and any $f \in \mathcal{S}(\mathbb{R}^2)$,*

$$\left\|\left(\partial^j \varphi_{\rho,m}\right) f\right\|_\kappa \le C\left\{\left(\frac{m}{\rho}\right)^j \|f\|_\kappa + \left(\frac{m}{\rho}\right)^{j+\kappa}\|f\|_0\right\}, \quad 0 \le j \le 2. \tag{2.10}$$

We prove now Theorem 2.1 by the following Proposition.

Proposition 2.6. *Let $w \in C^\infty(\bar{B})$ be a smooth solution of the quasi-linear equation (2.1). Suppose $k \in G^{\ell+1}(\mathbb{R}^2)$. Then there exists a constant L, such that for any integer $m \geq 5$, we have the following estimate*

$$\begin{aligned} &\|\varphi_{\rho,m}\partial^m w\|_{2+\frac{j}{\ell+1}} + \|\partial_s \Lambda^{2+\frac{j-1}{\ell+1}}\varphi_{\rho,m}\partial^m w\|_0 + \|\tilde{k}^{\frac{1}{2}}\partial_t \Lambda^{2+\frac{j-1}{\ell+1}}\varphi_{\rho,m}\partial^m w\|_0 \\ &\leq \frac{L^{m-2}}{\rho^{(\ell+1)(m-3)}}\left(\frac{m}{\rho}\right)^j \left((m-3)!\right)^{\ell+1},\ 0 \leq j \leq \ell+1,\ 0 < \rho < 1. \end{aligned} \tag{2.11}$$

Remark 2.7. *The constant L in Proposition 2.6 depends on $\ell, [w]_{8,\bar{B}}$, the Gevrey constant of k, and is independent of m.*

As an immediate consequence, for each compact subset $K \subset B$, if we choose $\rho_0 = \frac{1}{2}\text{dist}\,(K, \partial B)$. Then $\varphi_{\rho_0,m} = 1$ on K for any m, and (2.11) for $j = 0$ yields,

$$\left\|\partial^m w\right\|_{L^2(K)} \leq \left(\frac{L}{\rho_0^{\ell+1}}\right)^{m+1} (m!)^{\ell+1}, \quad \forall\, m \in \mathbb{N}.$$

This gives $u \in G^{\ell+1}(B)$. The proof of Theorem 2.1 is thus completed.

The proof of Proposition 2.6 is by induction on m. We state now the following two Lemmas, and postpone their proof to the last section.

Lemma 2.8. *Let $k \in G^{\ell+1}(\mathbb{R}^2)$ and $w \in C^\infty(\bar{B})$ be a solution of equation (2.1). Suppose that for some $N > 5$, (2.11) is satisfied for any $5 \leq m \leq N-1$, and that for some $0 \leq j_0 \leq \ell$, we have*

$$\begin{aligned} &\left\|\varphi_{\rho,N}\partial^N w\right\|_{2+\frac{j_0}{\ell+1}} + \left\|\partial_s \Lambda^{2+\frac{j_0-1}{\ell+1}}\varphi_{\rho,N}\partial^N w\right\|_0 + \left\|\tilde{k}^{\frac{1}{2}}\partial_t \Lambda^{2+\frac{j_0-1}{\ell+1}}\varphi_{\rho,N}\partial^N w\right\|_0 \\ &\leq \frac{C_0 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0}\left((N-3)!\right)^{\ell+1}, \quad \forall\, 0 < \rho < 1, \end{aligned} \tag{2.12}$$

where C_0 is a constant independent of L, N. Then there exists a constant C_1 independent of L, N, such that for any $0 < \rho < 1$,

$$\left\|\mathcal{L}\varphi_{\rho,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \leq \frac{C_1 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left((N-3)!\right)^{\ell+1}. \tag{2.13}$$

Lemma 2.9. *Let $k \in G^{\ell+1}(\mathbb{R}^2)$ and $w \in C^\infty(\bar{B})$ be a solution of equation (2.1). Suppose that for some $N > 5$, (2.11) is satisfied for any $5 \leq m \leq N-1$. Then there exists a constants C_2 independent of L, N, such that*

$$\left\|\mathcal{L}\varphi_{\rho,N}\partial^{N-1} w\right\|_{2+\frac{j-1}{\ell+1}} \leq \frac{C_2 L^{N-3}}{\rho^{(\ell+1)(N-4)}}\left(\frac{N-1}{\rho}\right)^{j+1}\left((N-4)!\right)^{\ell+1} \tag{2.14}$$

for all $0 < \rho < 1,\ 0 \leq j \leq \ell+1$.

Here and throughout the proof, C and C_j are used to denote suitable constants which depend on ℓ, $[\tilde{k}]_{0,B}$, $[w]_{8,\bar{B}}$ and the Gevrey constant of k, but it is independent of m and L.

Proof of Proposition 2.6. The proof is by induction on m. Firstly by using (2.10), the direct calculus implies, for $m = 5$, all $0 < \rho < 1$ and all integers j with $0 \leq j \leq \ell+1$,

$$\left\|\varphi_{\rho,m}\partial^m w\right\|_{2+\frac{j}{\ell+1}} + \left\|\partial_s \Lambda^{2+\frac{j-1}{\ell+1}}\varphi_{\rho,m}\partial^m w\right\|_0 + \left\|\tilde{k}^{\frac{1}{2}}\partial_t \Lambda^{2+\frac{j-1}{\ell+1}}\varphi_{\rho,m}\partial^m w\right\|_0 \leq \frac{M_1}{\rho^4}$$

with M_1 a constant depending only on $[\tilde{k}]_{0,B}, [w]_{8,\bar{B}}$ and the constant C in (2.10). Then (2.11) obviously holds for $m \le 5$ if we choose $L \ge M_1$.

Now we can finish the proof of Proposition 2.6 by induction, for any $N > 5$

Claim :*(2.11) is true for $m = N$ if it is true for all* $3 \le m \le N-1$.

We prove this claim again by induction on j, for $0 \le j \le \ell + 1$.
Case of $j = 0$: We apply Remark 2.4 with $r = 2 - \frac{1}{\ell+1}$, $m = 1$ and $v = \varphi_{\rho,N}\partial^{N-1}w \in C_0^\infty(B)$,

$$
\begin{aligned}
&\|\varphi_{\rho,N}\partial^N w\|_2^2 + \|\partial_s\Lambda^{2-\frac{1}{\ell+1}}\varphi_{\rho,N}\partial^N w\|_0^2 + \|\tilde{k}^{\frac12}\partial_t\Lambda^{2-\frac{1}{\ell+1}}\varphi_{\rho,N}\partial^N w\|_0^2\\
&\le \|\varphi_{\rho,N}\partial^{N-1}w\|_{2+1}^2 + \|\partial_s\Lambda^{2-\frac{1}{\ell+1}}\partial^1(\varphi_{\rho,N}\partial^{N-1}w)\|_0^2\\
&\quad + \|\tilde{k}^{\frac12}\partial_t\Lambda^{2-\frac{1}{\ell+1}}\partial^1(\varphi_{\rho,N}\partial^{N-1}w)\|_0^2 + C\left\|(\partial^1\varphi_{\rho,N})\partial^{N-1}w\right\|_{3-\frac{1}{\ell+1}}^2\\
&\le C_3\left\{\left\|\mathcal{L}\varphi_{\rho,N}\partial^{N-1}w\right\|_{3-\frac{2}{\ell+1}}^2 + \|\varphi_{\rho,N}\partial^{N-1}w\|_0^2 + \left\|\left(\partial^1\varphi_{\rho,N}\right)\partial^{N-1}w\right\|_{3-\frac{1}{\ell+1}}^2\right\}.
\end{aligned}
$$

By the induction assumption, we use now Lemma 2.9, to get

$$
\begin{aligned}
\left\|\mathcal{L}\varphi_{\rho,N}\partial^{N-1}w\right\|_{3-\frac{2}{\ell+1}} &= \left\|\mathcal{L}\varphi_{\rho,N}\partial^{N-1}w\right\|_{2+\frac{\ell-1}{\ell+1}}\\
&\le \frac{C_2L^{N-3}}{\rho^{(\ell+1)(N-4)}}\left(\frac{N-1}{\rho}\right)^{\ell+1}\left[\,(N-4)!\right]^{\ell+1}\\
&\le \frac{2^{\ell+1}C_2L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left[\,(N-3)!\right]^{\ell+1}.
\end{aligned}
$$

Hence the proof will be complete if we can show that (the term $\|\varphi_{\rho,N}\partial^{N-1}w\|_0$ is easier to treat)

$$
\left\|\left(\partial^1\varphi_{\rho,N}\right)\partial^{N-1}w\right\|_{3-\frac{1}{\ell+1}} \le \frac{C_4L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left[\,(N-3)!\right]^{\ell+1}. \tag{2.15}
$$

Setting $\rho_1 = \frac{(N-1)\rho}{N}$, then for any $k \ge 2$,

$$
\varphi_{\rho_1,k} = 1, \quad \text{on} \qquad B_{\rho_1},
$$

which implies that $\varphi_{\rho_1,k} = 1$ on the Supp $\varphi_{\rho,N} \subset B_{\rho_1}$ for any $k \ge 2$. From (2.10), we have

$$
\begin{aligned}
&\left\|\left(\partial^1\varphi_{\rho,N}\right)\partial^{N-1}w\right\|_{3-\frac{1}{\ell+1}} = \left\|\left(\partial^1\varphi_{\rho,N}\right)\varphi_{\rho_1,N-1}\partial^{N-1}w\right\|_{2+\frac{\ell}{\ell+1}}\\
&\le C_5\left\{\left(\frac{N}{\rho}\right)\left\|\varphi_{\rho_1,N-1}\partial^{N-1}w\right\|_{2+\frac{\ell}{\ell+1}} + \left(\frac{N}{\rho}\right)^{4-\frac{1}{\ell+1}}\left\|\varphi_{\rho_1,N-1}\partial^{N-1}w\right\|_0\right\}.
\end{aligned}
$$

On the other hand, the induction assumption with $m = N-1$, $j = \ell$, $0 \le \rho_1 \le 1$, yields

$$
\begin{aligned}
\frac{N}{\rho}\|\varphi_{\rho_1,N-1}\partial^{N-1}w\|_{2+\frac{\ell}{\ell+1}} &\le \frac{N}{\rho}\frac{L^{N-3}}{\rho_1{}^{(\ell+1)(N-4)}}\Big(\frac{N-1}{\rho_1}\Big)^{\ell}\left[\,(N-4)!\right]^{\ell+1}\\
&\le (2e)^{\ell+1}\frac{L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left[\,(N-3)!\right]^{\ell+1}.
\end{aligned}
$$

Setting now $\tilde{\rho}_1 = \frac{(N-2)\rho_1}{N-1}$, then for any $k \ge 2$,

$$
\varphi_{\tilde{\rho}_1,k} = 1, \quad \text{on} \qquad B_{\tilde{\rho}_1},
$$

which implies that $\varphi_{\tilde\rho_1,k} = 1$ on the Supp $\varphi_{\rho_1,N} \subset B_{\tilde\rho_1}$ for any $k \geq 2$. The induction assumption with $m = N-3$, $j = 0, 0 \leq \tilde\rho_1 \leq 1$, yields

$$\begin{aligned}\left(\frac{N}{\rho}\right)^{4-\frac{1}{\ell+1}} \left\|\varphi_{\rho_1,N-1}\partial^{N-1}w\right\|_0 &= \left(\frac{N}{\rho}\right)^{4-\frac{1}{\ell+1}} \left\|\varphi_{\rho_1,N-1}\partial^2\varphi_{\tilde\rho_1,N-3}\partial^{N-3}w\right\|_0\\ &\leq \left(\frac{N}{\rho}\right)^{4-\frac{1}{\ell+1}} \left\|\varphi_{\tilde\rho_1,N-3}\partial^{N-3}w\right\|_2\\ &\leq \left(\frac{N}{\rho}\right)^{4-\frac{1}{\ell+1}} \frac{L^{N-5}}{\tilde\rho_1^{(\ell+1)(N-6)}}\big[\,(N-6)!\big]^{\ell+1}\\ &\leq C_\ell\frac{L^{N-5}}{\rho^{(\ell+1)(N-3)}}\big[\,(N-3)!\big]^{\ell+1},\end{aligned}$$

where we have used the fact that

$$3(\ell+1)-4+\frac{1}{\ell+1} \geq 0, \quad \forall\ \ell \geq 0.$$

Therefore, we get (2.15) with $C_4 = C_5((2e)^{\ell+1} + 2C_\ell$, and finally for all $0 < \rho < 1$,

$$\begin{aligned}&\left\|\varphi_{\rho,N}\partial^N w\right\|_2 + \left\|\partial_s\Lambda^{2-\frac{1}{\ell+1}}\varphi_{\rho,N}\partial^N w\right\|_0 + \left\|\tilde k^{\frac12}\partial_t\Lambda^{2-\frac{1}{\ell+1}}\varphi_{\rho,N}\partial^N w\right\|_0\\ &\qquad\leq \frac{L^{N-2}}{\rho^{(\ell+1)(N-3)}}\big[\,(N-3)!\big]^{\ell+1},\end{aligned}\tag{2.16}$$

if we choose

$$L \geq 2C_3^{1/2}(2^{\ell+1}C_2 + C_4).$$

We prove now that (2.11) is true for $m = N$ **and** $j = j_0+1$ **if it is true for** $m = N$ **and** $0 \leq j \leq j_0$. We apply (2.4) with $r = 2+\frac{j_0}{\ell+1}$ and $v = \varphi_{\rho,N}\partial^N w \in C_0^\infty(B)$,

$$\begin{aligned}&\|\varphi_{\rho,N}\partial^N w\|^2_{2+\frac{j_0+1}{\ell+1}} + \|\partial_s\Lambda^{2+\frac{j_0}{\ell+1}}\varphi_{\rho,N}\partial^N w\|_0^2 + \|\tilde k^{\frac12}\partial_t\Lambda^{2+\frac{j_0}{\ell+1}}\varphi_{\rho,N}\partial^N w\|_0^2\\ &\leq C_3\left\{\left\|\mathcal{L}\varphi_{\rho,N}\partial^N w\right\|^2_{2+\frac{j_0-1}{\ell+1}} + \|\varphi_{\rho,N}\partial^N w\|_0^2\right\}.\end{aligned}$$

Firstly,

$$\begin{aligned}\|\varphi_{\rho,N}\partial^N w\|_0^2 \leq \|\varphi_{\rho_1,N-2}\partial^{N-2}w\|_2 &\leq \frac{L^{N-4}}{{\rho_1}^{(\ell+1)(N-5)}}\Big((N-5)!\Big)^{\ell+1}\\ &\leq e^{2(\ell+1)}\frac{L^{N-4}}{\rho^{(\ell+1)(N-5)}}\Big((N-5)!\Big)^{\ell+1}.\end{aligned}$$

Now for the term $\|\mathcal{L}\varphi_{\rho,N}\partial^N w\|_{2+\frac{j_0-1}{\ell+1}}$, we are exactly in the hypothesis of Lemma 2.8, (2.13) implies that

$$\begin{aligned}&\|\varphi_{\rho,N}\partial^N w\|_{2+\frac{j_0+1}{\ell+1}} + \|\partial_s\Lambda^{2+\frac{j_0}{\ell+1}}\varphi_{\rho,N}\partial^N w\|_0 + \|\tilde k^{\frac12}\partial_t\Lambda^{2+\frac{j_0}{\ell+1}}\varphi_{\rho,N}\partial^N w\|_0\\ &\leq C_3^{1/2}\frac{(C_1+e^{\ell+1})L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\Big((N-3)!\Big)^{\ell+1}.\end{aligned}$$

Finally, if we choose

$$L \geq \max\Big\{M_1,\ 2C_3^{1/2}(2^{\ell+1}C_2+C_4),\ C_3^{1/2}(C_1+e^{\ell+1})\Big\},$$

we get the validity of (2.11) for $j = j_0 + 1$, and hence for all $0 \le j \le \ell + 1$. Thus the proof of Proposition 2.6 is completed. □

3. Gevrey regularity of solutions for Monge-Ampère equations

In this section we prove Theorem 1.1. In the following discussions, we always assume $u(x, y)$ is a smooth solution of the Monge-Ampère equation (1.2) and $u_{yy} > 0$ in Ω, a neighborhood of the origin.

We first introduce the classic partial Legendre transformation (e.g. [16]) to translate the Gevrey regularity problem to the divergence form quasi-linear equation (2.1). Define the transformation $T : (x, y) \longrightarrow (s, t)$ by setting

$$\left\{ \begin{array}{lcl} s & = & x, \\ t & = & u_y. \end{array} \right. \tag{3.1}$$

It is easy to verify that

$$J_T = \begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u_{xy} & u_{yy} \end{pmatrix},$$

and

$$J_T^{-1} = \begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{u_{xy}}{u_{yy}} & \frac{1}{u_{yy}} \end{pmatrix}.$$

Thus if $u \in C^\infty$ and $u_{yy} > 0$ in Ω, then the transformations

$$T : \Omega \longrightarrow T(\Omega), \quad T^{-1} : T(\Omega) \longrightarrow \Omega$$

are C^∞ diffeomorphism. In [9], P. Guan proved that if $u(x, y) \in C^{1,1}(\Omega)$ is a weak solution of the Monge-Ampère equation (1.2) and $u_{yy} > 0$ in Ω, then $y(s, t) \in C^{0,1}(T(\Omega))$ is a weak solution of equation

$$\partial_{ss} y + \partial_t \Big\{ k(s, y(s, t)) \partial_t y \Big\} = 0. \tag{3.2}$$

He proved also the smoothness of $y(s, t) \in C^\infty(T(\Omega))$ and $u \in C^\infty(\Omega)$.

We prove now the following theorem which, together with Theorem 2.1, implies immediately Theorem 1.1.

Theorem 3.1. *Let $y(s, t) \in G^{\ell+1}(T(\Omega))$ be a solution of equation (3.2). Assume that $k(x, y) \in G^{\ell+1}(\Omega)$. Then $u(x, y) \in G^{\ell+1}(\Omega)$.*

We begin with the following results, which can be found in Rodino's book [14] (page 21).

Lemma 3.2. *If $g(z), h(z) \in G^{\ell+1}(U)$, then $(g\,h)(z) \in G^{\ell+1}(U)$, and moreover $\frac{1}{g(z)} \in G^{\ell+1}(U)$ if $g(z) \neq 0$. If $H \in G^{\ell+1}(\Omega)$ and the mapping $v : U \longrightarrow \Omega$ is $G^{\ell+1}(U)$, then $H(v(\cdot)) \in G^{\ell+1}(U)$.*

We study now the stability of Gevrey regularity by non linear composition. The following result is due to Friedman [7].

Lemma 3.3 (Lemma 1 of [7])**.** *Let M_j be a sequence of positive numbers satisfying the following monotonicity condition:*

$$\frac{j!}{i!(j-i)!} M_i M_{j-i} \le C^* M_j, \ (i = 1, 2, \cdots, j; \ j = 1, 2, \cdots) \tag{3.3}$$

with C^* a constant. Let $F(z,p)$ be a smooth function defined on $\Omega\times(-b,b)\subset\mathbb{R}^2\times\mathbb{R}$ satisfying that, for some constant C,

$$\max_{(z,p)\in\Omega\times(-b,b)}\left|\partial_z^\gamma\partial_p^i F(z,p)\right|\leq C^{|\gamma|+i}M_{|\gamma|-2}M_{i-2},$$

for all $\gamma\in\mathbb{Z}_+^2$, $i\in\mathbb{Z}_+$ with $|\gamma|, i\geq 2$. Then there exist two constants $\tilde{C}, C_*$, depending only on the above constants C^* and C, such that for every $H_0, H_1>1$ with $H_1\geq\tilde{C}H_0$, if the smooth function $\xi(z)$ satisfies that $\max_{z\in\Omega}|\xi(z)|<b$ and that

$$\max_{z\in B}\left|\partial_z^\beta\xi(z)\right|\leq H_0,\quad \textit{for } \beta \textit{ with } |\beta|\leq 1, \tag{3.4}$$

$$\max_{z\in B}\left|\partial_z^\beta\xi(z)\right|\leq H_0H_1^{|\beta|-2}M_{|\beta|-2},\quad \textit{for all } \beta\in\mathbb{Z}_+^2 \textit{ with } 2\leq|\beta|\leq N, \tag{3.5}$$

where $N\geq 2$ is a given integer, then for all $\alpha\in\mathbb{Z}_+^2$ with $|\alpha|=N$,

$$\max_{z\in B}\left|\partial_z^\alpha\left(F(z,\xi(z))\right)\right|\leq C_*H_0H_1^{N-2}M_{N-2}.$$

Remark 3.4. *Under the same assumptions as the above lemma, if we replace (3.4) and (3.5), respectively, by*

$$\max_{z\in\Omega}\left|\partial_{z_i}^m\xi(z)\right|\leq H_0,\quad \textit{for } m\leq 1,$$
$$\max_{z\in\Omega}\left|\partial_{z_i}^m\xi(z)\right|\leq H_0H_1^{m-2}M_{m-2},\quad \textit{for all } m\in\mathbb{Z}_+ \textit{ with } 2\leq m\leq N,$$

with $1\leq i\leq 2$ some fixed integer and $N\geq 2$ a given integer, then

$$\max_{z\in\Omega}\left|\partial_{z_i}^N\left(F(z,\xi(z))\right)\right|\leq C_*H_0H_1^{N-2}M_{N-2}.$$

We prepare firstly two propositions. In the follows, let K be any fixed compact subset of Ω.

Proposition 3.5. *Assume that $y(s,t)\in G^{\ell+1}(T(\Omega))$ and $k(x,y)\in G^{\ell+1}(\Omega)$, then the functions $F_m(s,t)\in G^{\ell+1}\left(T(\Omega)\right), m=1,2,3,$ where*

$$F_1(s,t)=(u_{xy}\circ T^{-1})(s,t)=u_{xy}(x(s,t),y(s,t)),$$
$$F_2(s,t)=(u_{xx}\circ T^{-1})(s,t)=u_{xx}(x(s,t),y(s,t)),$$

and

$$F_3(s,t)=(u_{yy}\circ T^{-1})(s,t)=u_{yy}(x(s,t),y(s,t)).$$

Proof. Indeed, since $y(s,t)\in G^{\ell+1}\left(T(\Omega)\right)$, then we conclude $y_s(s,t), y_t(s.t)\in G^{\ell+1}\left(T(\Omega)\right)$; that is

$$-\frac{u_{xy}(x(s,t),y(s,t))}{u_{yy}(x(s,t),y(s,t))},\ \frac{1}{u_{yy}(x(s,t),y(s,t))}\in G^{\ell+1}\left(T(\Omega)\right).$$

Lemma 3.2 yields that $F_3(s,t), F_1(s,t)\in G^{\ell+1}\left(T(\Omega)\right)$. Moreover, the fact that $k(x,y)\in G^{\ell+1}(\Omega)$ and $x(s,t)=s, y(s,t)\in G^{\ell+1}\left(T(\Omega)\right)$, implies $k(s,y(s,t))\in G^{\ell+1}\left(T(\Omega)\right)$, we have, in view of the equation (1.2),

$$F_2(s,t)=u_{xx}(x(s,t),y(s,t))\in G^{\ell+1}\left(T(\Omega)\right).$$

This gives the conclusion. ◻

As a consequence of the above proposition, there exists a constant M_*, depending only on the Gevrey constants of $k(x,y)$ and $y(s,t)$, such that for all $i, j \in \mathbb{Z}_+$ with $i, j \geq 2$,

$$\max_{(s,t)\in T(K)} \left|\partial_s^i \partial_t^j F_m(s,t)\right| \leq M_*^{i+j}\left[\,(i-2)!\right]^{\ell+1}\left[\,(j-2)!\right]^{\ell+1}, \quad m = 1,2,3. \tag{3.6}$$

Proposition 3.6. *Assume that $y(s,t) \in G^{\ell+1}(T(\Omega))$ and $k(x,y) \in G^{\ell+1}(\Omega)$. There exists a constant $\mathcal{M}$, depending only on the Gevrey constants of the functions $y(s,t)$ and $k(x,y)$, such that for all $i \geq 2$,*

$$\max_{(x,y)\in K} \left|\partial_x^i u_y(x,y)\right| + \max_{(x,y)\in K} \left|\partial_x^i u_x(x,y)\right| \leq 2[u]_{3,K}\mathcal{M}^{i-2}[(i-2)!]^{\ell+1}. \tag{3.7}$$

Proof. We first use induction on integer i to show that

$$\max_{(x,y)\in K} \left|\partial_x^i u_y(x,y)\right| \leq [u]_{3,K}\mathcal{M}^{i-2}[(i-2)!]^{\ell+1}, \quad i \geq 2. \tag{3.8}$$

Obviously, (3.8) is valid for $i = 2$. Now assuming

$$\max_{(x,y)\in K} \left|\partial_x^i u_y(x,y)\right| \leq [u]_{3,K}\mathcal{M}^{i-2}((i-2)!)^{\ell+1}, \quad \text{for all } 2 \leq i \leq N \tag{3.9}$$

with $N \geq 2$ an integer, we need to show that

$$\max_{(x,y)\in K} \left|\partial_x^{N+1} u_y(x,y)\right| = \max_{(x,y)\in K} \left|\partial_x^N u_{xy}(x,y)\right| \leq [u]_{3,K}\mathcal{M}^{N-1}[(N-1)!]^{\ell+1}. \tag{3.10}$$

Observe that $F_1 = u_{xy} \circ T^{-1}$ which implies

$$u_{xy}(x,y) = (F_1 \circ T)(x,y) = F_1(x, u_y(x,y)).$$

Thus the desired estimate (3.10) will follow if we can prove

$$\max_{(x,y)\in K} \left|\partial_x^N [F_1\left(x, u_y(x,y)\right)]\right| \leq [u]_{3,K}\mathcal{M}^{N-1}[(N-1)!]^{\ell+1}. \tag{3.11}$$

In the following we shall apply Remark 3.4 to deduce the above estimate.

Define

$$M_j = (j!)^{\ell+1}, \quad H_0 = [u]_{3,K}, \quad H_1 = \mathcal{M}.$$

Clearly $\left\{M_j\right\}$ satisfies the monotonicity condition (3.3). Furthermore, (3.9) and (3.6) yield

$$\max_{(x,y)\in K} \left|\partial_x^i u_y(x,y)\right| \leq H_0, \quad \text{for } i \leq 1,$$
$$\max_{(x,y)\in K} \left|\partial_x^i u_y(x,y)\right| \leq H_0 H_1^{i-2} M_{i-2}, \quad \text{for all } i \text{ with } 2 \leq i \leq N,$$

and

$$\max_{(s,t)\in T(K)} \left|\partial_s^i \partial_t^j F_1(s,t)\right| \leq M_*^{i+j} M_{i-2} M_{j-2}, \quad \text{for all } i, j \in \mathbb{N} \text{ with } i, j \geq 2.$$

Then it follows from the above three inequalities that the conditions in Remark 3.4 are satisfied, with $z_i = x, \xi(z) = u_y(x,y)$ and $F(z,\xi(z)) = F_1(x, u_y(x,y))$. This yields

$$\begin{aligned}\max_{(x,y)\in K} \left|\partial_x^N [F_1\left(x, u_y(x,y)\right)]\right| &\leq C_* H_0 H_1^{N-2} M_{N-2} \\ &= C_*[u]_{3,K}\mathcal{M}^{N-2}[(N-2)!]^{\ell+1}\end{aligned}$$

with C_* a constant depending only on M_* and hence on the Gevrey constants of $y(s,t)$ and $k(x,y)$. Then estimate (3.11) follows if we choose $\mathcal{M}$ large enough such that $\mathcal{M} \geq C_*$. This completes the proof of (3.8).

Now it remains to prove

$$\max_{(x,y)\in K}\left|\partial_x^i u_x(x,y)\right| \le [u]_{3,K}\mathcal{M}^{i-2}\left[(i-2)!\right]^{\ell+1}, \quad i\ge 2.$$

This can be deduce similarly as above. In view of (3.8) and (3.6), we can use Remark 3.4, with $z=(x,y), z_i=x, \xi(z)=u_y(x,y)$ and $F(z,\xi(z))=F_2(x,u_y(x,y))$, to obtain the above estimate. □

End of the proof of Theorem 3.1: Now we can show $u\in G^{\ell+1}(\Omega)$, i.e.,

$$\max_{(x,y)\in K}\left|\partial^\alpha u(x,y)\right| \le 2[u]_{3,K}\mathcal{M}^{m-3}\left[(m-3)!\right]^{\ell+1}, \quad \forall\ |\alpha|=m\ge 3, \tag{3.12}$$

where $\mathcal{M}$ is the constant given in (3.7).

We use induction on m. The validity of (3.12) for $m=3$ is obvious. Assuming, for some positive integer $m_0\ge 4$,

$$\max_{(x,y)\in K}\left|\partial^\gamma u(x,y)\right| \le 2[u]_{3,K}\mathcal{M}^{m-3}\left[(m-3)!\right]^{\ell+1}, \quad \forall\ 3\le|\gamma|=m\le m_0-1. \tag{3.13}$$

we need to show the validity of (3.12) for $m=m_0$. In the following discussions, let α be any fixed multi-index with $|\alpha|=m_0$. In view of (3.7), we only need to consider the case when $\partial^\alpha=\partial^{\tilde\alpha}\partial_y^2$ with $\tilde\alpha$ a multi-index satisfying $|\tilde\alpha|=m_0-2$. Observe $F_3=u_{yy}\circ T^{-1}$ which implies

$$u_{yy}(x,y)=(F_3\circ T)(x,y)=F_3(x,u_y(x,y)).$$

Hence

$$\partial^\alpha u=\partial^{\tilde\alpha}u_{yy}=\partial^{\tilde\alpha}[F_3(x,u_y(x,y))], \quad |\tilde\alpha|=m_0-2.$$

So the validity of (3.12) for $m=m_0$ will follow if we show that, for any $|\tilde\alpha|=m_0-2$,

$$\max_{(x,y)\in K}\left|\partial^{\tilde\alpha}[F_3(x,u_y(x,y))]\right| \le 2[u]_{3,K}\mathcal{M}^{m_0-3}\left[(m_0-3)!\right]^{\ell+1}. \tag{3.14}$$

To obtain the above estimate, we take M_j, H_0, H_1 as in the proof of Proposition 3.6; that is

$$M_j=(j!)^{\ell+1}, \quad H_0=[u]_{3,K}, \quad H_1=\mathcal{M}.$$

Then from (3.6) and the induction assumption (3.13), one has

$$\max_{(x,y)\in K}\left|\partial^\gamma u_y(x,y)\right|\le 2H_0, \quad \text{for } |\gamma|=m\le 1,$$
$$\max_{(x,y)\in K}\left|\partial^\gamma u_y(x,y)\right|\le 2H_0H_1^{m-2}M_{m-2}, \quad \text{for all } 2\le|\gamma|=m\le m_0-2,$$

and

$$\max_{(s,t)\in T(K)}\left|\partial_s^i\partial_t^j F_3(s,t)\right|\le M_*^{i+j}M_{i-2}M_{j-2}, \quad \text{for all } i,j\in\mathbb{N} \text{ with } i,j\ge 2.$$

Consequently, Lemma 3.3, with $z=(x,y), \xi(z)=u_y(x,y), N=m_0-2$ and $F(z,\xi(z))=F_3(x,u_y(x,y))$, yields for any $|\tilde\alpha|=m_0-2$

$$\begin{aligned}\max_{(x,y)\in K}\left|\partial^{\tilde\alpha}\left[F_3\left(x,u_y(x,y)\right)\right]\right| &\le 2\tilde C H_0H_1^{m_0-4}M_{m_0-4}\\ &=2\tilde C[u]_{3,K}\mathcal{M}^{m_0-4}\left[(m_0-4)!\right]^{\ell+1},\end{aligned}$$

where $\tilde C$ is a constant depending only the Gevrey constants of $k(x,y)$ and $y(s,t)$. Thus (3.14) follows if we choose $\mathcal{M}$ large enough such that $\mathcal{M}\ge 2\tilde C$. This gives validity of (3.12) for $m=m_0$ and hence for all $m\ge 3$, completing the proof of Theorem 3.1.

4. Technical lemmas

In this section, we prove the technical Lemmas(Lemma 2.8 and Lemma 2.9) used in the section 2. Firstly as an analogue of Lemma 3.3, we have

Lemma 4.1. *Let $N > 4$ and $0 < \rho < 1$ be given. Let $\{M_j\}$ be a positive sequence satisfying the monotonicity condition (3.3) and that*

$$M_j \geq \rho^{-j}, \quad j \geq 0.$$

Suppose $F(s,t,p), g(s,t)$ are two smooth functions satisfying the following two conditions:

1) There exists a constant C such that for any $j, l \geq 2$,

$$\left\|\partial_{s,t}^{\gamma}\partial_p^l F\right\|_{C^4(\bar{B}\times[-b,b])} \leq C^{j+l}M_{j-2}M_{l-2}, \quad \forall\ |\gamma| = j,$$

where $b = [g]_{0,\bar{B}}$ and $\|\cdot\|_{C^4(\bar{B}\times[-b,b])}$ is the standard Hörder norm.

2) There exist two constants $H_0, H_1 \geq 1$, satisfying $H_1 \geq \tilde{C}H_0$ with $\tilde{C}$ a constant depending only on the above constant C, such that $[g]_{6,\bar{B}} \leq H_0$ and for any $0 < \rho_* < 1$ with $\rho_* \approx \rho$ and any $j, 2 \leq j \leq N$,

$$\|\varphi_{\rho_*,j}\partial^j g\|_\nu \leq H_0 H_1^{j-2} M_{j-2},$$

where $1 < \nu < 4$ is a real number.

Then there exists a constant C_* depending only on C, such that

$$\left\|\varphi_{\rho,N}\partial^N\left(F(\cdot, g(\cdot))\right)\right\|_\nu \leq C_* H_0 H_1^{N-2} M_{N-2}.$$

Proof. The proof is similar to Lemma 5.3 of [3], so we give only main idea of the proof here. In the proof, we use C_n to denote constants which depend only on n and may be different in different contexts. By Faà di Bruno' formula, $\varphi_{\rho,N}D^\alpha[F(\cdot, g(\cdot))]$ is the linear combination of terms of the form

$$\varphi_{\rho,N}\left(\partial_{s,t}^{\beta}\partial_p^l F\right)(\cdot, g(\cdot)) \cdot \prod_{j=1}^{l}\partial^{\gamma_j} g, \tag{4.1}$$

where $|\beta| + l \leq |\alpha|$ and $\gamma_1 + \gamma_2 + \cdots + \gamma_l = \alpha - \beta$, and if $\gamma_i = 0$, $D^{\gamma_i}g$ doesn't appear in (4.1). Since $H^\nu(\mathbb{R}^2)$ for $\nu > 1$ is an algebra, then we have

$$\begin{aligned}
&\|\varphi_{\rho,N}\left(\partial_{s,t}^{\beta}\partial_p^l F\right)(\cdot, g(\cdot)) \cdot \prod_{j=1}^{l}\partial^{\gamma_j} g\|_\nu \\
&\leq \left\|\psi\left(\partial_{s,t}^{\beta}\partial_p^l F\right)(\cdot, g(\cdot))\right\|_\nu \cdot \prod_{j=1}^{l}\left\|\varphi_{\rho,|\gamma_j|}\partial^{\gamma_j} g\right\|_\nu,
\end{aligned}$$

where $\psi \in C_0^\infty(\mathbb{R}^2)$ and $\psi = 1$ on supp $\varphi_{\rho,N}$. The above inequality allows us to adopt the approach used by Friedman to prove Lemma 3.3, to get the desired estimate. Instead of the L^∞ norm in Lemma 3.3, we use H^ν-norm here. But there is no additional difficulty since $H^\nu(\mathbb{R}^2)$ is an algebra. We refer to [7] for more detail. $\square$

Applying the above result to the functions $\tilde{k}(s,t) \overset{\text{def}}{=} k(s, w(s,t))$ and $\tilde{k}_w(s,t) \overset{\text{def}}{=} k_y(s, w(s,t))$, we have

Corollary 4.2. *Let $N_0 > 4$ and $j_0 \in [0, \ell + 1]$ be any given integers. Suppose $k(x,y) \in G^{\ell+1}(\mathbb{R}^2)$ and $w(s,t) \in C^\infty(\bar{B})$ satisfying that for all $5 \leq m \leq N_0$ and all ρ with $0 < \rho < 1$,*

$$\left\|\varphi_{\rho,m}\partial^m w\right\|_{2+\frac{j_0-1}{\ell+1}} \leq \frac{c_* L^{m-2}}{\rho^{(\ell+1)(m-3)}}\left(\frac{m}{\rho}\right)^{j_0}\left[\,(m-3)!\right]^{\ell+1}, \tag{4.2}$$

where L, c_* are two constants with c_* independent of L. Then there exists a constant $\tilde{c}$, depending only on the Gevrey constants of k, w, and the above constant c_*, such that for all $5 \le m \le N_0$ and all ρ with $0 < \rho < 1$,

$$\begin{aligned}\left\|\varphi_{\rho,m}\partial^m \tilde{k}\right\|_{2+\frac{j_0-1}{\ell+1}} + \left\|\varphi_{\rho,m}\partial^m \tilde{k}_w\right\|_{2+\frac{j_0-1}{\ell+1}} \\ \le \frac{\tilde{c}L^{m-2}}{\rho^{(\ell+1)(m-3)}}\left(\frac{m}{\rho}\right)^{j_0}\left[\,(m-3)!\right]^{\ell+1}.\end{aligned} \tag{4.3}$$

Proof. We set $H_0 = c_*\left([w]_{8,\bar{B}}+1\right), H_1 = L$ and

$$M_0 = \frac{1}{\rho^3}, \quad M_j = \frac{[(j-1)!]^{\ell+1}}{\rho^{(\ell+1)(j-1)}}\left(\frac{j+2}{\rho}\right)^{j_0}, \quad j \ge 1.$$

Then by (4.2), we have

$$\left\|\varphi_{\rho,m}\partial^m w\right\|_{2+\frac{j_0-1}{\ell+1}} \le H_0 H_1^{m-2} M_{m-2}, \quad 2 \le m \le N_0.$$

On the other hand, the fact that $k \in G^{\ell+1}(\mathbb{R}^2)$, $k_y \in G^{\ell+1}(\mathbb{R}^2)$ and $M_j \ge [(j-1)!]^{\ell+1}$ implies

$$\left\|\partial_x^i\partial_y^j k(x,y)\right\|_{C^4(\Omega)} + \left\|\partial_x^i\partial_y^j k_y(x,y)\right\|_{C^4(\Omega)} \le C^{i+j}M_{i-2}M_{j-2}, \quad \forall\, i, j \ge 2,$$

where C is the Gevrey constant of k. Then by Lemma 4.1, the desired inequality (4.3) will follow if we show that $\{M_j\}$ satisfies the monotonicity condition (3.3). For every $0 < i < j$, we compute

$$\begin{aligned}\binom{j}{i}M_iM_{j-i} &= \frac{j!}{i!(j-i)!}\frac{((i-1)!)^{\ell+1}}{\rho^{(\ell+1)(i-1)}}\left(\frac{i+2}{\rho}\right)^{j_0}\frac{\left((j-i-1)!\right)^{\ell+1}}{\rho^{(\ell+1)(j-i-1)}}\left(\frac{j-i+2}{\rho}\right)^{j_0} \\ &= \frac{1}{\rho^{(\ell+1)(j-2)}}\frac{j!((i-1)!)^{\ell}((j-i-1)!)^{\ell}}{i(j-i)}\left(\frac{i+2}{\rho}\right)^{j_0}\left(\frac{j-i+2}{\rho}\right)^{j_0} \\ &\le \frac{9^{j_0}}{\rho^{(\ell+1)(j-2)}}\frac{j!((j-2)!)^{\ell}}{i(j-i)}\left(\frac{i}{\rho}\right)^{j_0}\left(\frac{j-i}{\rho}\right)^{j_0} \\ &\le \left\{9^{\ell+1}\rho^{(\ell+1)-j_0}\frac{j^2 i^{j_0-1}(j-i)^{j_0-1}}{(j-1)^{\ell+1}(j+2)^{j_0}}\right\}\frac{((j-1)!)^{\ell+1}}{\rho^{(\ell+1)(j-1)}}\left(\frac{j+2}{\rho}\right)^{j_0} \\ &\le \left\{9^{\ell+1}\rho^{(\ell+1)-j_0}\frac{j^2 j^{2(j_0-1)}}{(j-1)^{\ell+1}(j+2)^{j_0}}\right\}\frac{((j-1)!)^{\ell+1}}{\rho^{(\ell+1)(j-1)}}\left(\frac{j+2}{\rho}\right)^{j_0} \\ &\le C_\ell M_j,\end{aligned}$$

where C_ℓ is a constant depending only on ℓ. In the last inequality we used the fact that $\ell + 1 - j_0 \ge 0$. This completes the proof of Corollary 4.2. □

We prove now the technical Lemmas of section 2. We present a complete proof of Lemma 2.8, but omit the proof of Lemma 2.9 since it is similar.

Proof of Lemma 2.8. We recall the hypothesis of Lemma 2.8; that is, one has

(1) $k \in G^{\ell+1}(\mathbb{R}^2)$ and $\mathcal{L}w = 0$;

(2) for some $N > 5$, (2.11) is satisfied for any $5 \le m \le N-1$;

(3) for some $0 \le j_0 \le \ell$,

$$\begin{aligned} &\left\|\varphi_{\rho,N}\partial^N w\right\|_{2+\frac{j_0}{\ell+1}} + \left\|\partial_s \Lambda^{2+\frac{j_0-1}{\ell+1}}\varphi_{\rho,N}\partial^N w\right\|_0 + \left\|\tilde{k}^{\frac{1}{2}}\partial_t \Lambda^{2+\frac{j_0-1}{\ell+1}}\varphi_{\rho,N}\partial^N w\right\|_0 \\ &\qquad \le \frac{C_0 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0}\Big((N-3)!\Big)^{\ell+1}. \end{aligned} \tag{4.4}$$

We want to prove

$$\left\|\mathcal{L}\varphi_{\rho,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \le \frac{C_1 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\Big((N-3)!\Big)^{\ell+1} \tag{4.5}$$

for all $0 < \rho < 1$.

It follows from $\mathcal{L}w = 0$ that

$$\mathcal{L}\varphi_{\rho,N}\partial^\alpha w = [\mathcal{L},\ \varphi_{\rho,N}]\partial^\alpha w + \varphi_{\rho,N}[\mathcal{L},\ \partial^\alpha]w, \quad |\alpha| = N.$$

Hence the desired estimate (4.5) will follow if we can prove that

$$\left\|[\mathcal{L},\ \varphi_{\rho,N}]\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \le \frac{C_1 L^{N-3}}{2\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\big[(N-3)!\big]^{\ell+1}, \tag{4.6}$$

and

$$\sum_{|\alpha|=N}\left\|\varphi_{\rho,N}[\mathcal{L},\ \partial^\alpha]w\right\|_{2+\frac{j_0-1}{\ell+1}} \le \frac{C_1 L^{N-3}}{2\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\big[(N-3)!\big]^{\ell+1}. \tag{4.7}$$

We shall proceed to show the above two estimates by the following steps. As a convention, in the sequel we use C_j to denote different constants independent of L, N.

Step 1. We claim

$$\left\|\varphi_{\rho,m}\partial^m w\right\|_0 \le \frac{C_1 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{-2(\ell+1)}\big[(N-3)!\big]^{\ell+1}, \quad \forall\, 3 \le m \le N. \tag{4.8}$$

To confirm this, we set $\tilde{\rho} = \frac{(m-1)\rho}{m}$. Then

$$\left\|\varphi_{\rho,m}\partial^m w\right\|_0 = \left\|\varphi_{\rho,m}\partial^2\varphi_{\tilde{\rho},m-2}\partial^{m-2}u\right\|_0 \le \left\|\varphi_{\tilde{\rho},m-2}\partial^{m-2}u\right\|_2,$$

we can use (2.11) with $j = 0$ to compute

$$\begin{aligned} \left\|\varphi_{\tilde{\rho},m-2}\partial^{m-2}u\right\|_2 &\le \frac{L^{(m-2)-2}}{\tilde{\rho}^{(\ell+1)((m-2)-3)}}\big[((m-2)-3)!\big]^{\ell+1} \\ &\le \frac{C_0 L^{N-4}}{\rho^{(\ell+1)(m-5)}}\big[(m-5)!\big]^{\ell+1} \\ &\le \left(\frac{N}{\rho}\right)^{-2(\ell+1)}\frac{C_0 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\big[(N-3)!\big]^{\ell+1}, \end{aligned}$$

which implies (4.8) at once.

Step 2. In this step, we shall prove the following two inequalities:

$$\left\|\big(\partial_t\varphi_{\rho,N}\big)\tilde{k}\partial_t\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \le \frac{C_2 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\big[(N-3)!\big]^{\ell+1} \tag{4.9}$$

and

$$\left\|\big(\partial_s\varphi_{\rho,N}\big)\partial_s\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \le \frac{C_3 L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\big[(N-3)!\big]^{\ell+1}. \tag{4.10}$$

To prove the first inequality (4.9), we use (2.10) to get

$$
\begin{aligned}
&\left\|(\partial_t\varphi_{\rho,N})\tilde{k}\partial_t\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} = \left\|(\partial_t\varphi_{\rho,N})\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}}\\
&\leq C_4\left\{\left(\frac{N}{\rho}\right)\left\|\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} + \left(\frac{N}{\rho}\right)^{3+\frac{j_0-1}{\ell+1}}\left\|\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_0\right\}.
\end{aligned}
$$

Furthermore, the interpolation inequality (2.3) gives

$$
\begin{aligned}
&\left(\frac{N}{\rho}\right)^{3+\frac{j_0-1}{\ell+1}}\left\|\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_0\\
&\leq \left(\frac{N}{\rho}\right)\left\|\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} + \left(\frac{N}{\rho}\right)^{4+\frac{j_0-1}{\ell+1}}\left\|\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_{-1}\\
&\leq \left(\frac{N}{\rho}\right)\left\|\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} + C_5\left(\frac{N}{\rho}\right)^{4+\frac{j_0-1}{\ell+1}}\left\|\varphi_{\rho_1,N}\partial^N w\right\|_0\\
&\leq \left(\frac{N}{\rho}\right)\left\|\tilde{k}\partial_t\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} + \frac{C_6L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left[(N-3)!\right]^{\ell+1},
\end{aligned}
$$

where we have used (4.8) and $\Lambda^{-1}\tilde{k}\partial_t$ is bounded in L^2. On the other hand, note that

$$
\begin{aligned}
\left\|\tilde{k}\partial_t\varphi_{\tilde{\rho},N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} &\leq \left\|\tilde{k}\partial_t\Lambda^{2+\frac{j_0-1}{\ell+1}}\varphi_{\tilde{\rho},N}\partial^N w\right\|_0 + \left\|[\tilde{k},\ \Lambda^{2+\frac{j_0-1}{\ell+1}}]\partial_t\varphi_{\tilde{\rho},N}\partial^N w\right\|_0\\
&\leq C_7\left\{\left\|\tilde{k}^{\frac{1}{2}}\partial_t\Lambda^{2+\frac{j_0-1}{\ell+1}}\varphi_{\tilde{\rho},N}\partial^N w\right\|_0 + \left\|\varphi_{\tilde{\rho},N}\partial^N w\right\|_{2+\frac{j_0}{\ell+1}}\right\},
\end{aligned}
$$

which together with (4.4) yields:

$$
\left\|\tilde{k}\partial_t\varphi_{\tilde{\rho},N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \leq \frac{C_8L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0}\left[(N-3)!\right]^{\ell+1},
$$

and hence we obtain the desired inequality (4.9), combining the above inequalities. Similar arguments can be applied to prove (4.10). This completes the proof.

Step 3. We now claim that

$$
\begin{aligned}
&\left\|\left(\partial_{ss}\varphi_{\rho,N}\right)\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} + \left\|\left(\partial_{tt}\varphi_{\rho,N}\right)\tilde{k}\,\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}}\\
&\qquad\leq \frac{C_9L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left[(N-3)\right]^{\ell+1}.
\end{aligned}
\tag{4.11}
$$

To confirm this, we use (2.10) to get

$$
\begin{aligned}
&\left\|\left(\partial_{ss}\varphi_{\rho,N}\right)\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} + \left\|\left(\partial_{tt}\varphi_{\rho,N}\right)\tilde{k}\ \partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}}\\
&\leq C_{10}\left\{\left(\frac{N}{\rho}\right)^2\left\|\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} + \left(\frac{N}{\rho}\right)^{4+\frac{j_0-1}{\ell+1}}\left\|\varphi_{\rho_1,N}\partial^N w\right\|_0\right\}.
\end{aligned}
$$

The interpolation inequality (2.3) yields

$$\left(\frac{N}{\rho}\right)^2 \left\|\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \leq \left(\frac{N}{\rho}\right)\left\|\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0}{\ell+1}} + \left(\frac{N}{\rho}\right)^{2(\ell+1)+j_0+1} \left\|\varphi_{\rho_1,N}\partial^N w\right\|_0 .$$

The above two inequalities, together with (4.4) and (4.8), give the desired estimate (4.11) at once.

Step 4. Now we are ready to prove (4.6), the estimate on the commutator of $\mathcal{L}$ with the cut-off function $\varphi_{\rho,N}$. Firstly, one has

$$\begin{aligned}[\mathcal{L},\ \varphi_{\rho,N}] =& 2\left(\partial_s\varphi_{\rho,N}\right)\partial_s + \left(\partial_{ss}\varphi_{\rho,N}\right) + 2\left(\partial_t\varphi_{\rho,N}\right)\tilde{k}\partial_t \\ &+ \left(\partial_{tt}\varphi_{\rho,N}\right)\tilde{k} + \left(\partial_t\varphi_{\rho,N}\right)\left(\partial_t\tilde{k}\right).\end{aligned}$$

Observe that

$$\begin{aligned}\left\|\left(\partial_t\varphi_{\rho,N}\right)\left(\partial_t\tilde{k}\right)\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \leq & C_{11}\Big\{\left(\frac{N}{\rho}\right)\left\|\varphi_{\rho_1,N}\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \\ &+ \left(\frac{N}{\rho}\right)^{3+\frac{j_0-1}{\ell+1}} \left\|\varphi_{\rho_1,N}\partial^N w\right\|_0\Big\},\end{aligned}$$

hence from (4.4) and (4.8), we have

$$\left\|\left(\partial_t\varphi_{\rho,N}\right)\left(\partial_t\tilde{k}\right)\partial^N w\right\|_{2+\frac{j_0-1}{\ell+1}} \leq \frac{C_{12}L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}[(N-3)]^{\ell+1}.$$

Together with (4.9), (4.10) and (4.11), this yields the desired estimate (4.6) at once.

Step 5. In this step we shall deal with the non linear terms, and prove

$$\left\|\varphi_{\rho,N}\partial_t\partial^N\tilde{k}\right\|_{2+\frac{j_0-1}{\ell+1}} \leq \frac{C_{13}L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}[(N-3)!]^{\ell+1}. \tag{4.12}$$

Recall $\left\|\varphi_{\rho,N}\partial_t\partial^N\tilde{k}\right\|_{2+\frac{j_0-1}{\ell+1}} = \sum_{|\alpha|=N}\left\|\varphi_{\rho,N}\partial_t\partial^\alpha\tilde{k}\right\|_{2+\frac{j_0-1}{\ell+1}}$. Leibniz's formula gives, for any α with $|\alpha| = N$,

$$\begin{aligned}\varphi_{\rho,N}\partial_t\partial^\alpha\tilde{k} &= \sum_{1\leq|\beta|\leq|\alpha|}\binom{\alpha}{\beta}\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right) + \varphi_{\rho,N}\tilde{k}_w\partial_t\partial^\alpha w \\ &= \sum_{5\leq|\beta|\leq|\alpha|-4}\binom{\alpha}{\beta}\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right) + \varphi_{\rho,N}\tilde{k}_w\partial_t\partial^\alpha w + R_\alpha\end{aligned}$$

with $\tilde{k}_w(s,t) = k_w(s, w(s,t)))$ and

$$R_\alpha = \sum_{1\leq|\beta|\leq4}\binom{\alpha}{\beta}\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right) + \sum_{|\alpha|-3\leq|\beta|\leq|\alpha|}\binom{\alpha}{\beta}\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right).$$

Since $H^\kappa(\mathbb{R}^2), \kappa > 1$ is an algebra, we have

$$
\begin{aligned}
&\sum_{|\alpha|=N}\ \sum_{5\le|\beta|\le|\alpha|-4}\binom{\alpha}{\beta}\left\|\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right)\right\|_{2+\frac{j_0-1}{\ell+1}}\\
&\le \sum_{|\alpha|=N}\ \sum_{5\le|\beta|\le|\alpha|-4}\binom{\alpha}{\beta}\left\|\varphi_{\rho_1,|\beta|}\partial^\beta\tilde{k}_w\right\|_{2+\frac{j_0-1}{\ell+1}}\left\|\varphi_{\rho,N}\partial_t\partial^{\alpha-\beta}w\right\|_{2+\frac{j_0-1}{\ell+1}}\\
&\le \sum_{i=5}^{N-4}\frac{N!}{i!(N-i)!}\left\|\varphi_{\rho_1,i}\partial^i\tilde{k}_w\right\|_{2+\frac{j_0-1}{\ell+1}}\left\|\varphi_{\rho,N}\partial^{N-i+1}w\right\|_{2+\frac{j_0-1}{\ell+1}}.
\end{aligned}
$$

We can use (4.3) in Corollary 4.2, to get for each i with $5 \le i \le m$

$$
\begin{aligned}
\|\varphi_{\rho_1,i}\partial^i\tilde{k}_w\|_{2+\frac{j_0-1}{\ell+1}} &\le \frac{C_{14}L^{i-2}}{\rho_1{}^{(\ell+1)(i-3)}}\left(\frac{i}{\rho_1}\right)^{j_0}\left[\,(i-3)!\right]^{\ell+1}\\
&\le \frac{C_{15}L^{i-2}}{\rho^{(\ell+1)(i-3)}}\left(\frac{i}{\rho}\right)^{j_0}\left[\,(i-3)!\right]^{\ell+1}.
\end{aligned}
$$

Observing $N-i+1 \le N$ for each $i \ge 1$, we use (4.8) and the induction assumptions (2.11) and (4.4), to compute

$$
\begin{aligned}
&\left\|\varphi_{\rho,N}\partial^{N-i+1}w\right\|_{2+\frac{j_0-1}{\ell+1}} \le C\Big\{\left\|\varphi_{\rho_1,N-i+1}\partial^{N-i+1}w\right\|_{2+\frac{j_0-1}{\ell+1}}\\
&\qquad+\left(\frac{N}{\rho}\right)^{2+\frac{j_0-1}{\ell+1}}\left\|\varphi_{\rho_1,N-i+1}\partial^{N-i+1}w\right\|_0\Big\}\\
&\le \left\|\varphi_{\rho_1,N-i+1}\partial^{N-i+1}w\right\|_{2+\frac{j_0-1}{\ell+1}}+\frac{C_{16}L^{N-i-1}}{\rho^{(\ell+1)(N-i-2)}}\left(\frac{N-i+1}{\rho}\right)^{j_0}\left[(N-i-2)!\right]^{\ell+1}\\
&\le \left\|\varphi_{\rho_1,N-i+1}\partial^{N-i+1}w\right\|_{2+\frac{j_0}{\ell+1}}+\frac{C_{16}L^{N-i-1}}{\rho^{(\ell+1)(N-i-2)}}\left(\frac{N-i+1}{\rho}\right)^{j_0}\left[(N-i-2)!\right]^{\ell+1}\\
&\le \frac{C_{17}L^{N-i-1}}{\rho^{(\ell+1)(N-i-2)}}\left(\frac{N-i+1}{\rho}\right)^{j_0}\left[(N-i-2)!\right]^{\ell+1}.
\end{aligned}
$$

Then

$$\begin{aligned}
&\sum_{|\alpha|=N}\ \sum_{5\le|\beta|\le|\alpha|-4}\binom{\alpha}{\beta}\|\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right)\|_{2+\frac{j_0-1}{\ell+1}}\\
&\le \sum_{5\le i\le N-4}\frac{N!}{i!\,(N-i)!}\frac{C_{15}L^{i-2}}{\rho^{(\ell+1)(i-3)}}\left(\frac{i}{\rho}\right)^{j_0}\left[(i-3)!\right]^{\ell+1}\\
&\qquad\times\frac{C_{16}L^{N-i-1}}{\rho^{(\ell+1)(N-i-2)}}\left(\frac{N-i+1}{\rho}\right)^{j_0}\left[(N-i-2)!\right]^{\ell+1}\\
&\le \frac{C_{18}L^{N-3}}{\rho^{(\ell+1)(N-5)}}\left(\frac{N}{\rho}\right)^{2j_0}\sum_{5\le i\le N-3}\frac{N!}{i^3\,(N-i)^2}\left[(i-3)!\right]^{\ell}\left[(N-i-2)!\right]^{\ell}\\
&\le \frac{C_{18}L^{N-3}}{\rho^{(\ell+1)(N-4)}}\left(\frac{N}{\rho}\right)^{j_0+1}\sum_{5\le i\le N-4}\frac{(N-5)!N^{5+(j_0-1)}}{i^3\,(N-i)^2}\left[(N-5)!\right]^{\ell}\\
&\le \frac{C_{19}L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left[(N-3)!\right]^{\ell+1}\sum_{5\le i\le N-4}\frac{N^{4+j_0+1}}{N^{2(\ell+1)}i^3\,(N-i)^2}\\
&\le \frac{C_{19}L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left[(N-3)!\right]^{\ell+1}\sum_{5\le i\le N-4}\frac{N^2}{i^3\,(N-i)^2}.
\end{aligned}$$

Here the last inequality holds since $4+j_0-2(\ell+1)\le 2$. Moreover, observing that the series $\sum\limits_{5\le i\le N-4}\frac{N^2}{i^3(N-i)^2}$ is dominated from above by a constant independent of N, then we get

$$\begin{aligned}
&\sum_{|\alpha|=N}\ \sum_{5\le|\beta|\le|\alpha|-4}\binom{\alpha}{\beta}\|\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right)\|_{2+\frac{j_0-1}{\ell+1}}\\
&\le \frac{C_{20}L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left[(N-3)!\right]^{\ell+1}.
\end{aligned}$$

It’s a straightforward verification to prove that

$$\sum_{|\alpha|=N}\|R_\alpha\|_{2+\frac{j_0-1}{\ell+1}}\le\frac{C_{21}L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left[(N-3)!\right]^{\ell+1}.$$

So we have proved that

$$\begin{aligned}
&\sum_{|\alpha|=N}\ \sum_{1\le|\beta|\le|\alpha|}\binom{\alpha}{\beta}\left\|\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right)\right\|_{2+\frac{j_0-1}{\ell+1}}\\
&\le \frac{C_{22}L^{N-3}}{\rho^{(\ell+1)(N-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\left[(N-3)!\right]^{\ell+1}.
\end{aligned}\tag{4.13}$$

Observe $\left\|\varphi_{\rho,N}\partial_t\partial^N\tilde{k}\right\|_{2+\frac{j_0-1}{\ell+1}}$ is bounded from above by

$$\sum_{|\alpha|=N}\ \sum_{1\le|\beta|\le|\alpha|}\binom{\alpha}{\beta}\left\|\varphi_{\rho,N}(\partial^\beta\tilde{k}_w)\left(\partial_t\partial^{\alpha-\beta}w\right)\right\|_{2+\frac{j_0-1}{\ell+1}}+\|\varphi_{\rho,N}\tilde{k}_w\partial_t\partial^N w\|_{2+\frac{j_0-1}{\ell+1}}.$$

So to get the desired estimates (4.12) it remains to estimate the last term above. Direct calculations yield that

$$
\begin{aligned}
&\|\varphi_{\rho,N}\tilde{k}_w\partial_t\partial^N w\|_{2+\frac{j_0-1}{\ell+1}} = \|\varphi_{\rho,N}\tilde{k}_w\partial_t\varphi_{\rho_1,N}\partial^N w\|_{2+\frac{j_0-1}{\ell+1}}\\
&\leq C_{23}\Big\{\|\tilde{k}_w\Lambda^{2+\frac{j_0-1}{\ell+1}}\partial_t\varphi_{\rho_1,N}\partial^N w\|_0 + \|[\tilde{k}_w,\ \Lambda^{2+\frac{j_0-1}{\ell+1}}]\partial_t\varphi_{\rho_1,N}\partial^N w\|_0\\
&\qquad + \left(\frac{N}{\rho}\right)^{2+\frac{j_0-1}{\ell+1}}\|\tilde{k}_w\partial_t\varphi_{\rho_1,N}\partial^N w\|_0\Big\}\\
&\leq C_{24}\Big\{\|\tilde{k}_w\Lambda^{2+\frac{j_0-1}{\ell+1}}\partial_t\varphi_{\rho_1,N}\partial^N w\|_0 + \|\varphi_{\rho_1,N}\partial^N w\|_{2+\frac{j_0-1}{\ell+1}}\\
&\qquad + \left(\frac{N}{\rho}\right)^{2+\frac{j_0-1}{\ell+1}}\|\varphi_{\rho_1,N}\partial^N w\|_1\Big\}.\\
&\leq C_{25}\Big\{\|\tilde{k}_w\Lambda^{2+\frac{j_0-1}{\ell+1}}\partial_t\varphi_{\rho_1,N}\partial^N w\|_0 + \|\varphi_{\rho_1,N}\partial^N w\|_{2+\frac{j_0-1}{\ell+1}}\\
&\qquad + \left(\frac{N}{\rho}\right)^{\frac{(2\ell+j_0+1)^2}{(\ell+1)(j_0+\ell)}}\|\varphi_{\rho_1,N}\partial^N w\|_0\Big\}.
\end{aligned}
$$

In the last inequality we used the interpolation inequality (2.3). Combining the fact that

$$
|k_w(s,w)| \leq C\big(\sup_{w\in\mathbb{R}}|k_{ww}(s,w)|\,\big)^{\frac{1}{2}}\,(k(s,w))^{\frac{1}{2}}\,,
$$

which can be deduced from the nonnegativity of $k(s,w)$, we obtain

$$
\begin{aligned}
&\|\varphi_{\rho,N}\tilde{k}_w(s,w)\partial_t\partial^N w\|_{2+\frac{j_0-1}{\ell+1}}\\
&\leq C_{26}\Big\{\|\tilde{k}^{\frac{1}{2}}\Lambda^{2+\frac{j_0-1}{\ell+1}}\partial_t\varphi_{\rho_1,N}\partial^N w\|_0 + \|\varphi_{\rho_1,N}\partial^N w\|_{2+\frac{j_0}{\ell+1}}\\
&\qquad + \left(\frac{N}{\rho}\right)^{\frac{(2\ell+j_0+1)^2}{(\ell+1)(j_0+\ell)}}\|\varphi_{\rho_1,N}\partial^N w_0\Big\} \qquad (4.14)\\
&\leq \frac{C_{27}L^{|\alpha|-3}}{\rho^{(\ell+1)(|\alpha|-3)}}\left(\frac{N}{\rho}\right)^{j_0+1}\big[(|\alpha|-3)!\big]^{(\ell+1)},
\end{aligned}
$$

the last inequality following from (4.4) and (4.8). The proof is thus completed.

Step 6. Now we prepare to prove the inequality (4.7), the estimate on the commutator of $\mathcal{L}$ with the differential operator ∂^α. Direct verification yields

$$
[\mathcal{L},\ \partial^\alpha]w = -\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\left(\partial_t\partial^\beta\tilde{k}\right)\left(\partial_t\partial^{\alpha-\beta}w\right) - \sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\left(\partial^\beta\tilde{k}\right)\left(\partial_{tt}\partial^{\alpha-\beta}w\right).
$$

So

$$
\sum_{|\alpha|=N}\left\|\varphi_{\rho,N}[\mathcal{L},\ \partial^\alpha]w\right\|_{2+\frac{j_0-1}{\ell+1}} \leq \mathcal{S}_1 + \mathcal{S}_2 \qquad (4.15)
$$

with $\mathcal{S}_1, \mathcal{S}_2$ given by

$$
\mathcal{S}_1 = \sum_{|\alpha|=N}\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\left\|\varphi_{\rho,N}\left(\partial_t\partial^\beta\tilde{k}\right)\left(\partial_t\partial^{\alpha-\beta}w\right)\right\|_{2+\frac{j_0-1}{\ell+1}}
$$

and

$$\mathcal{S}_2 = \sum_{|\alpha|=N} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \left\| \varphi_{\rho,N} \left(\partial^{\beta}\tilde{k}\right) \left(\partial_{tt}\partial^{\alpha-\beta} w\right) \right\|_{2+\frac{j_0-1}{\ell+1}} .$$

For $\mathcal{S}_1$, we have treated the term of $\beta = \alpha$ by (4.12), and the terms of $0 < \beta < \alpha$ can be deduced similarly to (4.13); this gives

$$\mathcal{S}_1 \leq \frac{C_{28} L^{N-3}}{\rho^{(\ell+1)(N-3)}} \left(\frac{N}{\rho}\right)^{j_0+1} [(N-3)!]^{(\ell+1)}.$$

For $\mathcal{S}_2$, we have treated the term of $|\beta| = 1$ by (4.14), and the terms of $2 \leq |\beta| \leq |\alpha|$ can be deduced similarly to (4.13); this gives also

$$\mathcal{S}_2 \leq \frac{C_{29} L^{N-3}}{\rho^{(\ell+1)(N-3)}} \left(\frac{N}{\rho}\right)^{j_0+1} [(N-3)!]^{(\ell+1)}.$$

This complete the proof of Lemma 2.8. □

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